Reasoning with the finitely many-valued Łukasiewicz fuzzy Description Logic $S\text{ROIQ}$

Fernando Bobillo$^a$,*, Umberto Straccia$^b$

$^a$ Department of Computer Science and Systems Engineering, University of Zaragoza, Spain
$^b$ Istituto di Scienza e Tecnologie dell'Informazione (ISTI), Consiglio Nazionale delle Ricerche (CNR), Pisa, Italy

ABSTRACT

Fuzzy Description Logics are a formalism for the representation of structured knowledge affected by imprecision or vagueness. They have become popular as a language for fuzzy ontology representation. To date, most of the work in this direction has focused on the so-called Zadeh family of fuzzy operators (or fuzzy logic), which has several limitations. In this paper, we generalize existing proposals and show how to reason with a fuzzy extension of the logic $S\text{ROIQ}$, the logic behind the language OWL 2, under finitely many-valued Łukasiewicz fuzzy logic. We show for the first time that it is decidable over a finite set of truth values by presenting a reasoning preserving procedure to obtain a non-fuzzy representation for the logic. This reduction makes it possible to reuse current representation languages as well as currently available reasoners for ontologies.

© 2010 Elsevier Inc. All rights reserved.

1. Introduction

In the last years, the use of ontologies as formalisms for knowledge representation in many different application domains has grown significantly. Ontologies have been successfully used as part of expert and multiagent systems, as well as a core element in the Semantic Web, which proposes to extend the current web to give information a well-defined meaning [3].

An ontology is defined as an explicit and formal specification of a shared conceptualization [24], which means that ontologies represent the concepts and the relationships in a domain promoting interrelation with other models and automatic processing. Ontologies allow adding semantics to data, making knowledge maintenance, information integration, and reuse of components easier.

The current standard language for ontology creation is the Web Ontology Language (OWL [63]), which comprises three sublanguages of increasing expressive power: OWL Lite, OWL DL, and OWL Full. OWL Full is the most expressive level, but reasoning within it becomes undecidable; OWL Lite has the lowest complexity; and OWL DL is a balanced tradeoff between expressiveness and reasoning complexity. However, since its first development, several limitations on expressiveness of OWL have been identified, and consequently several extensions to the language have been proposed [55]. Among them, the most significant is OWL 2 [18], its most likely immediate successor which is currently a Proposed Recommendation at W3C [64].

Description Logics (DLs for short) [1] are a family of logics for representing structured knowledge. Each logic is denoted by using a string of capital letters which identify the constructors of the logic and therefore its complexity. DLs have proved to
be very useful as ontology languages [2]. For instance, OWL Lite, OWL DL and OWL 2 are close equivalents to SROIQ(D), SHOIN(D) and SROIQ(D), respectively [31].

Nevertheless, it has been widely pointed out that classical ontologies are not appropriate to deal with imprecise and vague knowledge, which is inherent to several real world domains [49].

Fuzzy logic is a suitable formalism to handle these types of knowledge. In the setting of fuzzy logics, the convention prescribing that a statement is either true or false is changed. A more refined range is used, in such a way that every fuzzy statement has a degree of truth $a \in [0,1]$ [28].

Several fuzzy extensions of DLs can be found in the literature (see [41] for a survey), as the theoretical basis of fuzzy ontologies. Fuzzy ontologies have proved to be useful in several applications, such as Chinese news summarization [39], semantic help-desk support [46], ontology-based query enrichment [37], information retrieval [15], or image interpretation, (e.g. recognition of brain structures in 3D magnetic resonance images [32]). There are also a lot of applications in the Semantic Web field (see for example [47,17]) and, more generally, in the Internet [49].

In fuzzy logic, all classical set operations are extended to the fuzzy case. The intersection, union, complement and implication set operations are performed by a t-norm function $\otimes$, a t-conorm function $\oplus$, a negation function $\ominus$, and an implication function $\Rightarrow$, respectively. These functions or fuzzy operators are grouped in families, also simply called fuzzy logics.

It is well known that different families of fuzzy operators lead to fuzzy DLs with different properties. There are three main fuzzy logics: Łukasiewicz, Gödel and Product. It is also common to consider the fuzzy set operators originally proposed by Zadeh: Gödel conjunction and disjunction, Łukasiewicz negation and Kleene–Dienes implication.

Although there has been a relatively significant amount of work in extending DLs with fuzzy set theory [41], most of the existing works restrict themselves to Zadeh fuzzy logic (see Section 2.4 for a definition and Section 5 for a detailed summary of the state of the art in fuzzy DLs).

This paper provides a reasoning algorithm for Łukasiewicz fuzzy SROIQ over a finite set of truth values, the logic behind OWL 2. This is the first reasoning algorithm for such an expressive logic, for which decidability was not known.

Compared to Zadeh logic, our proposal provides several advantages:

- Łukasiewicz fuzzy logic is more general than Zadeh fuzzy logic.
- The implication of Zadeh fuzzy logic (Kleene–Dienes implication) has some counter-intuitive effects [27,4]. For instance, a concept does not fully subsume itself. Łukasiewicz implication solves these problems.
- The t-norm of Zadeh and Gödel fuzzy logics (the minimum t-norm) is idempotent and hence it is not Pareto optimal [48]. This is problematic in some applications such as fuzzy matchmaking [48].

Defining a fuzzy DL brings about that standard languages would no longer be appropriate, new fuzzy languages should be used and hence the large number of resources available should be adapted to the new framework, requiring an important effort. An alternative is to represent fuzzy DLs using non-fuzzy DLs and to reason using these representations. Our reasoning algorithm will provide such a non-fuzzy representation.

The remainder of this work is organized as follows. Section 2 overviews some necessary background. Section 3 describes a fuzzy extension of SROIQ and particularizes it to the case of Łukasiewicz fuzzy logic. Section 4 depicts a reduction into SROIQ. Section 5 reviews some related work. Finally, Section 6 sets out some conclusions and ideas for future work.

2. Preliminaries

This section provides some basic background. Section 2.1 quickly overviews SROIQ [30], the DL which will be mainly treated throughout this paper. Then, Section 2.4 refreshes some basic ideas in mathematical fuzzy logic [28].

2.1. The Description Logic SROIQ

SROIQ extends ALC standard DL [52] with transitive roles (ALC plus transitive roles is called $S$), complex role axioms ($R$), nominals ($O$), inverse roles ($I$) and qualified number restrictions ($Q$).

2.2. Syntax

SROIQ assumes three alphabets of symbols, for concepts, roles and individuals. In DLs, complex concepts and roles can be built using different concept and role constructors. In SROIQ, the concepts (denoted $C$ or $D$) and roles ($R$) can be built inductively from atomic concepts ($A$), atomic roles ($R_A$), top concept $\top$, bottom concept $\bot$, named individuals ($o$), simple roles ($S$, which will be defined below) and universal role $U$, as shown in Table 1, where $n, m$ are natural numbers ($n \geq 0, m > 0$), $x, y \in \mathcal{D}$ are abstract individuals and $\#X$ denotes the cardinality of the set $X$.

Example 2.1. Man and Woman are atomic concepts. hasChild and likes are atomic roles. Man $\sqcap \top \geq 2$ hasChild. Woman is a complex concept representing a father with at least two daughters. 3likes.Since represents a narcissist.
A Knowledge Base (KB) comprises the intensional knowledge, i.e. general knowledge about the application domain (a Terminological Box or TBox $T$ and a Role Box or RBox $R$), and the extensional knowledge, i.e. particular knowledge about some specific situation (an Assertional Box or ABox $A$ with statements about individuals).

An ABox consists of a finite set of assertions about individuals:

- **Concept assertions** $a : C$, meaning that individual $a$ is an instance of $C$.
- **Role assertions** $(a, b) : R$, meaning that $(a, b)$ is an instance of $R$.
- **Negated role assertions** $(a, b) : \neg R$.
- **Inequality assertions** $a \neq b$.
- **Equality assertions** $a = b$.

A TBox consists of a finite set of general concept inclusion (GCI) axioms $C \sqsubseteq D$ ($C$ is more specific than $D$).

Let $w$ be a role chain (a finite string of roles not including the universal role $U$). An RBox consists of a finite set of role axioms:

- **Role inclusion axioms** (RIAs) $w \sqsubseteq R$ (role chain $w$ is more specific than $R$).
- **Transitive role axioms** $\text{trans}(R)$.
- **Disjoint role axioms** $\text{dis}(S_1, S_2)$.
- **Reflexive role axioms** $\text{ref}(R)$.
- **Irreflexive role axioms** $\text{irref}(S)$.
- **Symmetric role axioms** $\text{sym}(R)$.
- **Asymmetric role axioms** $\text{asy}(S)$.

**Example 2.2.** The concept assertion paul : Man states that the individual Paul belongs to the class of men. The role assertion (paul, John) : hasChild states that John is not the child of Paul. The GCI Man $\sqsubseteq$ Human states that all men are human. The RIA owns hasPart $\sqsubseteq$ owns states the fact if somebody owns something, he also owns its components.

Now we will introduce some definitions which will be useful to impose some limitations in the language. A **strict partial order** $\prec$ on a set $A$ is an irreflexive and transitive relation on $A$. A strict partial order $\prec$ on the set of roles is called a **regular order** if it satisfies $R_1 \prec R_2 \iff R_2 \prec R_1$, for all roles $R_1$ and $R_2$.

In order to guarantee the decidability of the logic, there are some restrictions in the use of roles. Given a regular order $\prec$, every role axiom cannot contain $U$ and every RIA should be $\prec$-regular. A RIA $w \sqsubseteq R$ is $\prec$-regular if $R$ is atomic and:

1. $w = RR$, or
2. $w = R^n$, or
3. $w = S_1, \ldots, S_n$ and $S_i \prec R$ for all $i = 1, \ldots, n$, or
4. $w = RS_1, \ldots, S_n$ and $S_i \prec R$ for all $i = 1, \ldots, n$, or
5. $w = S_1, \ldots, S_nR$ and $S_i \prec R$ for all $i = 1, \ldots, n$.

Note that, in order to prove decidability of the reasoning, roles are assumed to be simple in some concept constructors (local reflexivity, at-least and at-most number restrictions) and role axioms (disjoint, irreflexive and asymmetric role axioms) [30]. Simple roles are defined as follows:
1. $R_A$ is simple if it does not occur on the right side of a RIA.
2. $R^-$ is simple if $R$ is.
3. if $R$ occurs on the right side of a RIA, $R$ is simple if, for each $w \sqsubseteq R$, $w = S$ for a simple role $S$.

2.3. Semantics

An interpretation $I$ is a pair $(\mathcal{A}^I, \mathcal{I})$ consisting of a non-empty set $\mathcal{A}^I$ (the interpretation domain) and an interpretation function $\mathcal{I}$ mapping:

- Every individual $a$ onto an element $a^I$ of $\mathcal{A}^I$.
- Every atomic concept $A$ onto a set $A^I \subseteq \mathcal{A}^I$.
- Every atomic role $R_A$ onto a relation $R_A^I \subseteq \mathcal{A}^I \times \mathcal{A}^I$.

The interpretation is extended to complex concepts and roles by the inductive definitions in Table 1. Unique name assumption is not imposed, i.e. two nominals might refer to the same individual.

Let $\circ$ be the standard composition of relations. An interpretation $I$ satisfies (is a model of):

- $a : C$ if $a^I \in C^I$.
- $(a, b) : R$ if $(a^I, b^I) \in R^I$.
- $(a, b) : -R$ if $(a^I, b^I) \notin R^I$.
- $a \neq b$ if $a^I \neq b^I$.
- $a = b$ if $a^I = b^I$.
- $C \subseteq D$ if $C^I \subseteq D^I$.
- $R_1 \ldots R_n \subseteq R$ if $R_1 \circ \ldots \circ R_n \subseteq R^I$.
- $\text{trans}(R)$ if $(x, y) \in R^I$ and $(y, z) \in R^I \implies (x, z) \in R^I, \forall x, y, z \in \mathcal{A}^I$.
- $\text{dis}(S_1, S_2)$ if $S_1 \cap S_2 = \emptyset$.
- $\text{ref}(R)$ if $(x, x) \in R^I, \forall x \in \mathcal{A}^I$.
- $\text{irr}(S)$ if $(x, x) \notin S^I, \forall x \in \mathcal{A}^I$.
- $\text{sym}(S)$ if $(x, y) \in S^I \implies (y, x) \in S^I, \forall x \in \mathcal{A}^I$.
- $\text{asy}(S)$ if $(x, y) \notin S^I \implies (y, x) \notin S^I, \forall x \in \mathcal{A}^I$.
- $A$ Knowledge Base $K = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ if it satisfies each element in $\mathcal{A}, \mathcal{T}$ and $\mathcal{R}$.

A DL not only stores axioms and assertions, but also offers some reasoning services, such as KB satisfiability, concept satisfiability or subsumption. However, if a DL is closed under negation, most of the basic reasoning tasks are reducible to KB satisfiability [50], so it is usually the only task considered.

2.4. Mathematical fuzzy logic

In the setting of fuzzy logics, the convention prescribing that a statement is either true or false is changed. A more refined range is used for the function that represents the meaning of a statement. This is usual in natural language when words are modelled by fuzzy sets. For example, the compatibility of “tall” in the phrase “a tall man” with some individual of a given height is often graded: the man can be judged not quite tall, somewhat tall, rather tall, very tall, etc.

Changing the usual true/false convention leads to a new concept of statement, whose compatibility with a given state of facts is a matter of degree and can be measured on e.g., the unit interval $[0, 1]$. This degree of fit is called degree of truth of the statement $\phi$ in the interpretation $I$.

Fuzzy logics provide compositional calculi of degrees of truth, including degrees between “true” and “false”. A statement is now not true or false only, but may have a truth degree taken from a truth space $S$, usually $[0, 1]$ (in that case we speak about Mathematical Fuzzy Logic [28]) or $\{\frac{n-1}{n}, \ldots, \frac{1}{n}\}$ for an integer $n \geq 1$. Often $S$ may be also a complete lattice or a bilattice [23, 22].

In the illustrative fuzzy logic that we consider in this section, fuzzy statements have the form $\phi \geq \alpha$ or $\phi \leq \beta$, where $\alpha, \beta \in [0, 1]$ [26, 28] and $\phi$ is a statement, which encode that the degree of truth of $\phi$ is at least $l$ resp. at most $u$. For example, ripeTomato $\geq 0.9$ says that we have a rather ripe tomato (the degree of truth of ripeTomato is at least 0.9).

Semantically, a fuzzy interpretation $I$ maps each basic statement $p_i$ into $[0, 1]$ and is then extended inductively to all statements as follows:

\[
\begin{align*}
I(\phi \land \psi) &= I(\phi) \odot I(\psi); \\
I(\phi \lor \psi) &= I(\phi) \oplus I(\psi); \\
I(\phi \rightarrow \psi) &= I(\phi) \Rightarrow I(\psi); \\
I(\neg \phi) &= \ominus I(\phi),
\end{align*}
\] (1)
Table 2
Properties for t-norms and t-conorms.

<table>
<thead>
<tr>
<th>Axiom name</th>
<th>T-norm</th>
<th>S-norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tautology/contradiction</td>
<td>$x \otimes 0 = 0$</td>
<td>$x \otimes 1 = 1$</td>
</tr>
<tr>
<td>Identity</td>
<td>$x \otimes 1 = x$</td>
<td>$x \otimes 0 = x$</td>
</tr>
<tr>
<td>Commutativity</td>
<td>$x \otimes y = y \otimes x$</td>
<td>$x \oplus y = y \oplus x$</td>
</tr>
<tr>
<td>Associativity</td>
<td>$(x \otimes y) \otimes z = x \otimes (y \otimes z)$</td>
<td>$(x \oplus y) \oplus z = x \oplus (y \oplus z)$</td>
</tr>
<tr>
<td>Monotonicity</td>
<td>if $\beta \leq \gamma$, then $x \otimes \beta \leq x \otimes \gamma$</td>
<td>if $\beta \leq \gamma$, then $x \leq x \otimes \gamma$</td>
</tr>
</tbody>
</table>

Table 3
Properties for implication and negation functions.

<table>
<thead>
<tr>
<th>Axiom name</th>
<th>Implication function</th>
<th>Negation function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tautology/contradiction</td>
<td>$0 \Rightarrow \beta = 1$, $x \Rightarrow 1 = 1$, $\beta \Rightarrow 0 = \beta$</td>
<td>$0 \otimes 1 = 0, 1 \otimes 0 = 0$</td>
</tr>
<tr>
<td>Antitonicity</td>
<td>if $x \leq \beta$, then $\beta \Rightarrow x = \beta$</td>
<td>if $\beta \leq x$, then $\beta \Rightarrow x = \beta$</td>
</tr>
<tr>
<td>Monotonicity</td>
<td>if $\beta \leq \gamma$, then $\beta \Rightarrow \beta \leq \gamma$</td>
<td>if $\beta \leq \gamma$, then $\beta \Rightarrow \gamma$</td>
</tr>
</tbody>
</table>

where $\otimes$, $\oplus$, $\Rightarrow$, and $\otimes$ are so-called combination functions, namely, triangular norms (or t-norms), triangular co-norms (or t-conorms), implication functions, and negation functions, respectively, which extend the classical Boolean conjunction, disjunction, implication, and negation, respectively, to the fuzzy case.

Several t-norms, t-conorms, implication functions, and negation functions have been given in the literature. An important aspect of such functions is that they satisfy some properties that one expects to hold for the connectives; see Tables 2 and 3. Note that in Table 2, the two properties Tautology and Contradiction follow from Identity, Commutativity, and Monotonicity.

Some t-norms, t-conorms, implication functions, and negation functions of various fuzzy logics are shown in Table 4 [28].

In fuzzy logic, one usually distinguishes three different logics, namely, Łukasiewicz (denoted Ł), Gödel (denoted G), and Product logic (denoted PI). The importance of these three logics is due the fact that any continuous t-norm can be obtained as a combination of Łukasiewicz, Gödel, and Product t-norm [44].

The usually called “Zadeh logic” is a sublogic of Łukasiewicz fuzzy logic. Some salient properties of these four logics are shown in Table 5. For more properties, see especially [28,45]. Note also, that a fuzzy logic having all properties shown in Table 5, collapses to boolean logic, i.e. the truth-set can be $\{0,1\}$ only.

An involutive negation satisfies that $\odot(\ominus x) = x$. Łukasiewicz negation is involutive, while Gödel negation is not.

Usually, the implication function $\Rightarrow$ is defined as an R-implication, or the residuum of a t-norm $\otimes$, that is, $x \otimes \beta = \sup\{y : x \otimes y \leq \beta\}$. An S-implication is defined as $x \Rightarrow \beta = \ominus x \otimes \beta$. Łukasiewicz implication is both an R-implication and an S-implication. Gödel and Product logics have an R-implication, whereas Zadeh logic has an S-implication.

The implication $x \Rightarrow \beta = \max(1 - x, \beta)$ is called Kleeene–Dienes implication in the fuzzy logic literature. Note that we have the following inferences: Let $\phi \geq x$ and $\phi \Rightarrow \psi \geq \beta$. Then, under Kleene–Dienes implication, we infer that if $x > 1 - \beta$ then $\psi \geq \beta$. Under an R-implication relative to a t-norm $\otimes$, we infer that $\psi \geq x \otimes \beta$ instead.

Note that implication functions and t-norms are also used to define the degree of subsumption between fuzzy sets and the composition of two (binary) fuzzy relations. A fuzzy relation $R$ over a countable classical set $X$ is a function $R: X \times X \rightarrow [0,1]$. The degree of subsumption between two fuzzy sets $A$ and $B$, denoted $A \sqsubseteq B$, is defined as $\inf_{x \in X}(A(x) \Rightarrow B(x))$, where $\Rightarrow$ is an implication function. Note that if $A(x) \leq B(x)$, for all $x \in [0,1]$, then $A \sqsubseteq B$ evaluates to 1. Of course, $A \sqsubseteq B$ may evaluate to a value $x \in (0,1]$ as well.

A (binary) fuzzy relation $R$ over two countable classical sets $X$ and $Y$ is a function $R: X \times Y \rightarrow [0,1]$. The inverse of $R$ is the function $R^{-1}: Y \times X \rightarrow [0,1]$ with membership function $R^{-1}(y,x) = R(x,y)$, for every $x \in X$ and $y \in Y$. The composition of two fuzzy relations $R_1: X \times Y \rightarrow [0,1]$ and $R_2: Y \times Z \rightarrow [0,1]$ is defined as $(R_1 \circ R_2)(x, z) = \sup_{y \in Y}R_1(x, y) \otimes R_2(y, z)$. A fuzzy relation $R$ is transitive iff $R(x, z) \geq (R \circ R)(x, z)$.

A fuzzy interpretation $I$ satisfies a fuzzy statement $\phi \geq I$ (resp., $\phi \leq I$) or $I$ is a model of $\phi \geq I$ (resp., $\phi \leq I$), denoted $I \models \phi \geq I$ (resp., $I \models \phi \leq I$), iff $I(\phi) \geq I$ (resp., $I(\phi) \leq I$). The notions of satisfiability and logical consequence are defined in the standard way. We say that $\phi \geq I$ is a tight logical consequence of a set of fuzzy statements $\mathcal{K}$ iff $I$ is the infimum of $I(\phi)$ subject to all models $I$ of $\mathcal{K}$. Notice that the latter is equivalent to $I = \sup\{r | r \models \phi \geq r\}$. We refer the reader to [28] for reasoning algorithms for fuzzy propositional and First-Order Logics.

---

Note that $\oplus$ is also used in the context of Łukasiewicz logic to denote the binary connective $x \oplus y = \max(0, x - y)$. However, in this paper we will use it as a unary negation function.
3. Fuzzy SROIQ

In this section, we define a fuzzy extension of the DL SROIQ where concepts denote fuzzy sets of individuals and roles denote fuzzy binary relations. Axioms are also extended to the fuzzy case and some of them hold to a degree. The following definition is based on the fuzzy DLs presented in [56,4,53].

In the rest of the paper we will assume $\mathcal{P} = \{P, <, >\}$, $a \in (0,1]$, $b \in [0,1)$ and $c \in [0,1]$. The symmetric $\mathcal{C}^0$ and the negation $\neg$ of an operator $\mathcal{O}$ are defined as follows:

### 3.1. Syntax

Fuzzy SROIQ assumes three alphabets of symbols, for concepts, roles and individuals. Let $U$ be the universal role, and $R_A$ an atomic role. The roles of the language are built using the syntax rule:

$$ R \rightarrow R_A | U | R^- $$

(2)

The concepts of the language (denoted $C$ or $D$) can be built inductively from atomic concepts ($A$), top concept $\top$, bottom concept $\bot$, named individuals ($o_i$) and roles ($R$ and $S$, where $S$ is a simple role as defined below) according to the following syntax rule (with $n, m$ being natural numbers, $n \geq 0$, $m > 0$):

$$ C, D \rightarrow A | \top | \bot | C \sqcap D | C \cup D | \neg C \forall R.C | \exists R.C | \{x_1 \lleq o_1, \ldots , x_m \lleq o_m\} | (\lleq mS.C | (\lleq nS.C) | \exists S \self $$

(3)

The only difference with the non-fuzzy case is the presence of fuzzy nominals of the form $\{x_1 \lleq o_1, \ldots , x_m \lleq o_m\}$ [4]. We assume that $o_i \neq o_j$, $1 \leq i < j \leq m$.

**Example 3.1.** $\{1/\text{germany}, 1/\text{austria}, 0.67/\text{switzerland}\}$ represents the concept of German-speaking country, with Germany and Austria fully belonging to it, but Switzerland belonging only with degree 0.67, since only about two thirds of its population speak German.

---

**Table 4**

Combination functions of various fuzzy logics.

<table>
<thead>
<tr>
<th>Łukasiewicz fuzzy logic</th>
<th>Gödel logic</th>
<th>Product logic</th>
<th>Zadeh logic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a \odot b$</td>
<td>$\max(a+b-1,0)$</td>
<td>$\min(a,b)$</td>
<td>$a \odot b$</td>
</tr>
<tr>
<td>$a \odot b$</td>
<td>$\min(a+b,1)$</td>
<td>$\max(a,b)$</td>
<td>$a \odot b$</td>
</tr>
<tr>
<td>$a \Rightarrow b$</td>
<td>$\min(1-a+b,1)$</td>
<td>${1 \text{ if } a \leq b}$</td>
<td>$a \Rightarrow b$</td>
</tr>
<tr>
<td>$\ominus a$</td>
<td>$1-a$</td>
<td>${1 \text{ if } a = 0}$</td>
<td>$1-a$</td>
</tr>
</tbody>
</table>

---

**Table 5**

Some additional properties of combination functions of various fuzzy logics.

<table>
<thead>
<tr>
<th>Property</th>
<th>Łukasiewicz fuzzy logic</th>
<th>Gödel logic</th>
<th>Product logic</th>
<th>Zadeh logic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a \odot a = 0$</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>$a \odot a = 1$</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>$a \odot a = a$</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>$a \odot a = a$</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>$a \Rightarrow b \odot a \odot b$</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>$(a \Rightarrow b) \odot a \odot b$</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>$(a \Rightarrow b) \odot a \odot b$</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

---

2 We will also allow role negation in fuzzy assertions of the form $\langle a, b \rangle : \neg R \odot a$. 
A fuzzy KB $K$ comprises a fuzzy ABox $A$, a fuzzy TBox $T$ and a fuzzy RBox $R$.

A fuzzy ABox consists of a finite set of fuzzy assertions. A fuzzy assertion can be an inequality assertion $(a \neq b)$, an equality assertion $(a = b)$ or a constraint on the truth value of a concept or role assertion, i.e., an expression of the form $(\Psi \geq \beta)$, $(\Psi > \beta)$, $(\Psi < \beta)$ or $(\Psi < \xi)$, where $\Psi$ is of the form $a:C, (a, b):R$ or $(a, b):\neg R$.

A fuzzy TBox consists of fuzzy GCIs, which constrain the truth value of a GCI. They are expressions of the form $(C \subseteq D \geq \beta)$ or $(C \subseteq D > \beta)$.

A fuzzy RBox consists of a finite set of role axioms, which can be fuzzy RIAs $(w \subseteq R \geq \beta)$ or $(w \subseteq R > \beta)$ for a role chain $w = R_1R_2\ldots R_n$, or any other of the role axioms from the non-fuzzy case: transitive trans(R), disjoint dis(S_1, S_2), reflexive ref(R), irreflexive irref(R), symmetric sym(R) or asymmetric asym(R).

Example 3.2. The fuzzy concept assertion $(\text{paul: Tall} \geq 0.5)$ states that Paul is tall with at least degree 0.5. The fuzzy RIA $(\text{isFriendOf isFriendOf isFriendOf} \geq 0.75)$ states that the friends of my friends can also be considered as my friends with at least degree 0.75.

A fuzzy axiom is positive (denoted $(\tau \geq \beta)$) if it is of the form $(\tau \geq \beta)$ or $(\tau > \beta)$, and negative (denoted $(\tau < \omega)$) if it is of the form $(\tau < \beta)$ or $(\tau < \omega)$.

Notice that negative fuzzy GCIs or RIAs are not allowed, because they correspond to negated GCIs and RIAs, respectively, which are not part of SROIQ.

As in the non-fuzzy case, role axioms cannot contain $U$ and every RIA should be $\prec$-regular for a regular order $\prec$. A RIA $(w \subseteq R \gamma)$ is $\prec$-regular if $R$ is atomic and:

- $w = RR$, or
- $w = R^*$, or
- $w = S_1\ldots S_n$ and $S_i \prec R$ for all $i = 1,\ldots , n$, or
- $w = RS_1\ldots S_n$ and $S_i \prec R$ for all $i = 1,\ldots , n$, or
- $w = S_1\ldots S_n R$ and $S_i \prec R$ for all $i = 1,\ldots , n$.

Simple roles are defined as in the non-fuzzy case:

- $R_A$ is simple if it does not occur on the right side of a RIA.
- $R^*$ is simple if $R$ is.
- If $R$ occurs on the right side of a RIA, $R$ is simple if, for each $(w \subseteq R \gamma)$, $w = S$ for a simple role $S$.

3.2. Semantics

A fuzzy interpretation $I$ is a pair $(A^I, \tau^I)$ consisting of a non-empty set $A^I$ (the interpretation domain) and a fuzzy interpretation function $\tau^I$ mapping:

- Every individual $a$ onto an element $a^I$ of $A^I$.
- Every concept $C$ onto a function $C^I : A^I \rightarrow [0, 1]$.
- Every role $R$ onto a function $R^I : A^I \times A^I \rightarrow [0, 1]$.

$C^I$ (resp. $R^I$) denotes the membership function of the fuzzy concept $C$ (resp. fuzzy role $R$) w.r.t. $I$. $C^I(a)$ (resp. $R^I(a,b)$) gives us to what extent the individual $a$ can be considered as an element of the fuzzy concept $C$ (resp. to what extent $(a, b)$ can be considered as an element of the fuzzy role $R$) under the fuzzy interpretation $I$.

Given a t-norm $\otimes$, a t-conorm $\oplus$, a negation function $\ominus$ and an implication function $\Rightarrow$, the fuzzy interpretation function is extended to complex concepts and roles as follows:

\[
\uparrow^I(x) = 1 \\
\downarrow^I(x) = 0 \\
(C \cap D)^I(x) = C^I(x) \otimes D^I(x) \\
(C \cup D)^I(x) = C^I(x) \oplus D^I(x) \\
(-C)^I(x) = \ominus C^I(x) \\
(\forall R.C)^I(x) = \inf_{y \in A^I} \{R^I(x, y) \Rightarrow C^I(y)\} \\
(\exists R.C)^I(x) = \sup_{y \in A^I} \{R^I(x, y) \odot C^I(y)\} \\
\{a_1/o_1, \ldots, a_m/o_m\}^I(x) = \sup \{x | x = o^I_i\} 
\]
The following tasks can be reduced to fuzzy KB satisfiability:

Example 3.3.

We do not impose unique name assumption, i.e. two nominals might refer to the same individual. The fuzzy interpretation function is extended to fuzzy axioms as follows:

\[
(a : C)^\mathcal{I} = \mathcal{C}^\mathcal{I}(a^\mathcal{I})
\]

\[
((a, b) : R) = \mathcal{R}^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I})
\]

\[
((a, b) : \neg R^\mathcal{I}) = \mathcal{R}^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I})
\]

\[
(C \subseteq D) = \inf_{x \in \mathcal{A}} \{\mathcal{C}^\mathcal{I}(x) \Rightarrow \mathcal{D}^\mathcal{I}(x)\}
\]

\[
(R_1 \ldots R_n \subseteq R)^\mathcal{I} = \inf_{x_1, x_{n+1} \in \mathcal{A}} \left\{ \sup_{x_2, \ldots, x_n \in \mathcal{A}} \{ (R_1^\mathcal{I}(x_1, x_2) \otimes \ldots \otimes R_n^\mathcal{I}(x_n, x_{n+1})) \Rightarrow R^\mathcal{I}(x_1, x_{n+1})\} \right\}
\]

Note that this is the semantics for fuzzy RIs that is implicitly assumed in the non-fuzzy representations provided in [7,8].

A fuzzy interpretation \(\mathcal{I}\) satisfies (is a model of):

- \((a : C)^\mathcal{I} \models \gamma\) iff \((a : C)^\mathcal{I} \sqsupseteq \gamma\).
- \(((a, b) : R)^\mathcal{I} \models \gamma\) iff \((a, b) : R)^\mathcal{I} \sqsupseteq \gamma\).
- \((a \neq b) \models a^\mathcal{I} \neq b^\mathcal{I}\).
- \((a = b) \models a^\mathcal{I} = b^\mathcal{I}\).
- \((C \subseteq D)^\mathcal{I} \models \gamma\) iff \((C \subseteq D)^\mathcal{I} \sqsupseteq \gamma\).
- \((R_1, \ldots, R_n \subseteq R)^\mathcal{I} \models \gamma\) iff \((R_1, \ldots, R_n \subseteq R)^\mathcal{I} \sqsupseteq \gamma\).
- \(\text{trans}(R)\) iff \(\forall x, y, z \in \mathcal{A}^\mathcal{I}, R^\mathcal{I}(x, z) \otimes R^\mathcal{I}(z, y) \subseteq R^\mathcal{I}(x, y)\).
- \(\text{dis}(S_1, S_2)\) iff \(\forall x, y \in \mathcal{A}^\mathcal{I}, S_1^\mathcal{I}(x, y) = 0\) or \(S_2^\mathcal{I}(x, y) = 0\).
- \(\text{ret}(R)\) iff \(\forall x \in \mathcal{A}^\mathcal{I}, R^\mathcal{I}(x, x) = 1\).
- \(\text{irr}(S)\) iff \(\forall x \in \mathcal{A}^\mathcal{I}, S^\mathcal{I}(x, x) = 0\).
- \(\text{sym}(R)\) iff \(\forall x, y \in \mathcal{A}^\mathcal{I}, R^\mathcal{I}(x, y) = R^\mathcal{I}(y, x)\).
- \(\text{asy}(S)\) iff \(\forall x, y \in \mathcal{A}^\mathcal{I}, S^\mathcal{I}(x, y) > 0\) then \(S^\mathcal{I}(y, x) = 0\).
- a fuzzy KB \(\mathcal{K} = (\mathcal{A}, \mathcal{T}, \mathcal{R})\) iff it satisfies each element in \(\mathcal{A}, \mathcal{T}\) and \(\mathcal{R}\).

Notice that individual assertions are either true or false, as it has always been assumed in the fuzzy DL literature [56,54].

Given a fuzzy KB \(\mathcal{K}\), the problem of *fuzzy KB satisfiability* consists on checking the existence of a fuzzy interpretation satisfying \(\mathcal{K}\). In the rest of the paper we will only consider fuzzy KB satisfiability, since (as in the non-fuzzy case) most inference problems can be reduced to it [60].

**Example 3.3.** The following tasks can be reduced to fuzzy KB satisfiability:

- **Concept satisfiability.** \(C\) is \(\alpha\)-satisfiable w.r.t. a fuzzy KB \(\mathcal{K}\) iff \(\mathcal{K} \cup \{\alpha : C \sqsupseteq \alpha\}\) is satisfiable, where \(a\) is a new individual, which does not appear in \(\mathcal{K}\).
- **Entailment:** A fuzzy concept assertion \(a : C \sqsupseteq \alpha\) is entailed by a fuzzy KB \(\mathcal{K}\) (denoted \(\mathcal{K} \models \{a : C \sqsupseteq \alpha\}\)) iff \(\mathcal{K} \cup \{a : C \sqsupseteq \alpha\}\) is unsatisfiable. The case for fuzzy role assertions is similar.
- **Greatest lower bound.** The greatest lower bound of a concept or role assertion \(\tau\) is defined as the sup\{\(\alpha : \mathcal{K} \models \{\alpha : \tau \sqsupseteq \alpha\}\}\}. In Zadeh and infinitely many-valued Łukasiewicz and Gödel fuzzy logics, it can be computed performing several entailment tests.

3 More precisely, in finitely many-valued Łukasiewicz and Gödel fuzzy logics we need to assume a finite set of degrees of truth \(|\mathcal{V}|\) including 0 and 1 [27], and the number of tests is at most log\(|\mathcal{V}|\) [60].
Another important notion is that of witnessed interpretations. A fuzzy interpretation $I$ is witnessed [27] iff it verifies:

- $\forall x \in A^T$, there is $y \in A^T$ such that $(\exists R.C)^T(x) = R^T(x, y) \otimes C^T(y)$, and
- $\forall x \in A^T$, there is $y \in A^T$ such that $(\forall R.C)^T(x) = R^T(x, y) \Rightarrow C^T(y)$, and
- There is $x \in A^T$ such that $(C \sqsubseteq D)^T = C^T(x) \Rightarrow D^T(x)$, and
- There are $x_1, \ldots, x_{n+1} \in A^T$ such that $(R_1 \ldots R_n \sqsubseteq R)^T = (R_1^T(x_1, x_2) \otimes \cdots \otimes R_n^T(x_n, x_{n+1})) \Rightarrow R^T(x_1, x_{n+1})$, and
- If $I \models \text{trans}(R)$, for all $x, y \in A^T$, there is $z \in A^T$ such that $\sup_{z \in A^T} R^T(x, z) \otimes R^T(z, y) = R^T(x, z) \otimes R^T(z, y)$.

3.3. Łukasiewicz fuzzy logic

From now on we will concentrate on ŁSROIQ, restricting ourselves to the fuzzy operators of the Łukasiewicz fuzzy logic. It can be easily shown that ŁSROIQ is a sound extension of SROIQ, in the sense that fuzzy interpretations coincide with non-fuzzy interpretations if we restrict the degrees of truth to $\{0, 1\}$.

In Łukasiewicz logic, there are a lot of equivalences which allow the inter-definition of most of the concept constructors. The inter-definability of cardinality restrictions will be specially interesting for us. These equivalences are:

- $\neg(-C) \equiv C$.
- $\top \equiv \bot$.
- $\bot \equiv \top$.
- $C \sqcap D \equiv \neg(-C \sqcup \neg D)$.
- $C \sqcup D \equiv \neg(-C \sqcap \neg D)$.
- $\forall R.C \equiv \neg(\exists R, -C)$.
- $\exists R.C \equiv \neg(\forall R, -C)$.
- $(\geq m.S.C) \equiv \neg(\leq m - 1.S.C)$.
- $(\leq n.S.C) \equiv \neg(\geq n + 1.S.C)$.

The semantics of qualified cardinality restrictions has been proposed in [12] and verifies the following additional properties:

- If $(\leq n.R.C)^T(a) = 1$ then $|(b|(R(a, b)^T \otimes C(b))^T > 0)| \leq n$.
- $\exists S.C \equiv \geq 1.S.C$.

It has been shown that Łukasiewicz fuzzy logic verifies the Witnessed Model Property (WMP), i.e. for each countable model there is an equivalent witnessed model [27]. Hence, we can restrict ourselves to witnessed models.

In Łukasiewicz logic, there are several axioms which are syntactic sugar (and consequently it can be assumed that they do not appear in fuzzy KBs) due to the following equivalences:

**Proposition 3.4.** In ŁSROIQ, the following equivalences hold:

- $\langle (a, b) : \neg R \Rightarrow \gamma \rangle \equiv \langle (a, b) : R \Leftarrow 1 - \gamma \rangle$.
- $\text{irr}(S) \equiv \langle \top \sqsubseteq \neg S.S \Rightarrow 1 \rangle$.
- $\text{trans}(R) \equiv \langle R \sqsubseteq R \Rightarrow 1 \rangle$.
- $\text{sym}(R) \equiv \langle R \sqsubseteq R^\top \Rightarrow 1 \rangle$.

Finally, the finitely many-valued fragment of ŁSROIQ is denoted as Ł$_n$SROIQ, for some natural $n$.

4. A non-fuzzy representation for fuzzy Ł$_n$SROIQ

In this section we show how to reduce a Ł$_n$SROIQ fuzzy KB into a non-fuzzy KB whenever a finite truth space is assumed. We will start by presenting our reduction procedure, and then we will illustrate it with a couple of examples. Then we will discuss the properties of the reduction, showing that it preserves reasoning, so existing SROIQ reasoners could be applied to the resulting KB.

The basic idea behind the reduction procedure is to create some new non-fuzzy concepts and roles, representing the $\varepsilon$-cuts of the fuzzy concepts and relations, and to rely on them. Next, some new axioms are added to preserve their semantics and finally every axiom in the ABox, the TBox and the RBox is represented, independently from other axioms, using these new non-fuzzy elements.

4.1. Adding new elements

It has been shown that, for a fuzzy KB $K$ under Zadeh logic, the set of the degrees of truth which must be considered for any reasoning task is defined as $N^\gamma = X^\gamma \cup \{1 - \varepsilon | x \in X^\varepsilon \}$, where $X^\varepsilon = \{0, 0.5, 1\} \cup \{\gamma | (\tau \Rightarrow \gamma) \in K\}$ [61]. This holds for fuzzy
Proposition 4.1. Let \( \frac{n}{\alpha} \in \mathcal{N} \). Then, under the fuzzy operators of Łukasiewicz fuzzy logic, it is true if we fix the number of allowed degrees.

This result can be easily checked by considering the four fuzzy operators:

- \( \frac{n}{\alpha} \in \mathcal{N} \) belongs to \( \mathcal{N} \) if \( a \in [0, \alpha] \) and \( (n-a) \in [0, \alpha] \).
- \( \frac{n}{\alpha} \in \mathcal{N} \) is also true if \( a, b \in [0, \alpha] \) and \( n-a \in [0, \alpha] \).
- \( \frac{n}{\alpha} \in \mathcal{N} \) is given by the minimum and \( \min \) of the four values:
  - \( \frac{n}{\alpha} = \min(\frac{n}{\alpha} + \frac{a}{\alpha}, \frac{n}{\alpha} + \frac{b}{\alpha}) \) when \( \frac{n}{\alpha} + \frac{a}{\alpha} \in [0, \alpha] \) and \( \frac{n}{\alpha} + \frac{b}{\alpha} \in [0, \alpha] \).
- \( \frac{n}{\alpha} \in \mathcal{N} \) is given by the minimum and \( \min \) of the four values:
  - \( \frac{n}{\alpha} = \min(\frac{n}{\alpha} + \frac{a}{\alpha}, \frac{n}{\alpha} + \frac{b}{\alpha}) \) when \( \frac{n}{\alpha} + \frac{a}{\alpha} \in [0, \alpha] \) and \( \frac{n}{\alpha} + \frac{b}{\alpha} \in [0, \alpha] \).
- \( \frac{n}{\alpha} \in \mathcal{N} \) is given by the minimum and \( \min \) of the four values:
  - \( \frac{n}{\alpha} = \min(\frac{n}{\alpha} + \frac{a}{\alpha}, \frac{n}{\alpha} + \frac{b}{\alpha}) \) when \( \frac{n}{\alpha} + \frac{a}{\alpha} \in [0, \alpha] \) and \( \frac{n}{\alpha} + \frac{b}{\alpha} \in [0, \alpha] \).
- \( \frac{n}{\alpha} \in \mathcal{N} \) is given by the minimum and \( \min \) of the four values:
  - \( \frac{n}{\alpha} = \min(\frac{n}{\alpha} + \frac{a}{\alpha}, \frac{n}{\alpha} + \frac{b}{\alpha}) \) when \( \frac{n}{\alpha} + \frac{a}{\alpha} \in [0, \alpha] \) and \( \frac{n}{\alpha} + \frac{b}{\alpha} \in [0, \alpha] \).

Note that, given a new individual \( a \), and \( n \in \mathcal{N} \), we may always add a fuzzy assertion \( \langle a \rangle \) to a fuzzy KB without changing its meaning.

Now, we will assume that \( N^F = \mathcal{N} \) and proceed similarly as in [7], which creates an optimized number of new elements (concepts, roles and axioms) with respect to previous approaches.

Without loss of generality, it can be assumed that \( N^F = \{\gamma_1, \ldots, \gamma_{|\mathcal{N}|}\} \) and \( \gamma_i \leq \gamma_{i+1}, 1 \leq i \leq |N^F| - 1 \). It is easy to see that \( \gamma_1 = 0 \) and \( \gamma_{|\mathcal{N}|} = 1 \). We define \( N'' = \{x \in \mathcal{N} : x \neq 0\} \).

Let \( A \) and \( R \) be the set of fuzzy atomic concepts and fuzzy atomic roles occurring in a fuzzy KB \( \mathcal{C} = \langle A, T, R \rangle \), respectively.

For each \( \alpha, \beta \in \mathcal{N} \) with \( \alpha \in [0, 1] \) and \( \beta \in [0, 1] \), for each \( A \in A \), two new atomic concepts \( A_{\geq \alpha}, A_{\geq \beta} \) are introduced. \( A_{\geq \alpha} \) represents the (non-fuzzy) set of individuals which are instance of \( A \) with degree higher or equal than \( \alpha \) i.e. the \( \alpha \)-cut of \( A \). \( A_{\geq \beta} \) is defined in a similar way.

Similarly, for each \( R \in R \) two new atomic roles \( R_{\geq \alpha}, R_{\geq \beta} \). The atomic elements \( A_{\geq \alpha}, R_{\geq \alpha}, A_{\geq \beta} \) and \( R_{\geq \beta} \) are not considered because they are not necessary, due to the restrictions on the allowed degree of the axioms in the fuzzy KB (e.g. we do not allow GCIs of the form \( C \sqsubseteq D \).

The semantics of these newly introduced atomic concepts and roles is preserved by some terminological and role axioms. For each \( 1 \leq i \leq |\mathcal{N}| - 1, 2 \leq j \leq |\mathcal{N}| - 1 \) and for each \( A \in A, T(\mathcal{N}) \) is the smallest terminology containing these two axioms:

\[
A_{\geq \gamma_i} \sqsubseteq A_{\geq \gamma_i+1} \tag{4}
\]

\[
A_{\geq \gamma_i} \sqsubseteq A_{\geq \gamma_i+1} \tag{5}
\]

4.2. Mapping fuzzy concepts, roles and axioms

The reduction of concept and role expressions is achieved using a mapping \( \rho \). Given a fuzzy concept \( C, \rho(C, \geq \alpha) \) is a set containing all the elements which belong to \( C \) with a degree greater or equal than \( \alpha \). The other cases \( \rho(C, > \gamma) \) and \( \rho(R, > \gamma) \) for a role \( R \) are similar.

In Zadeh logic, due to the definition of the fuzzy operators, it is possible to infer exactly degrees of truth. For instance, given a expression of the form \( \rho(C \sqsubseteq D, > \alpha) \) we can infer both \( \rho(C, \geq \alpha) \) and \( \rho(D, \geq \alpha) \), since the semantics of the conjunction is given by the minimum and \( \min\{C^\alpha(x), D^\alpha(x)\} \geq \alpha \) clearly implies that \( C^\alpha(x) \geq \alpha \) and \( D^\alpha(x) \geq \alpha \).

In Łukasiewicz fuzzy logic, the situation is more complicated, since it is not possible to infer the exact degree of truth of the elements that compose a complex concept. However, thanks to Proposition 4.1, we know that they belong to \( \mathcal{N} \). Now, the basic idea of the reduction of fuzzy concepts is to build a disjunction over the different possible degrees of truth.

Before proceeding formally, we will illustrate this idea with an example.

Example 4.2. Consider a fuzzy assertion \( \tau = (a_1 A_1 \cap A_2 > 0.5) \) and \( \mathcal{N} = \{0.05, 0.5, 0.75, 1\} \). Every model \( I \) of \( \tau \) satisfies \( \max(A_1^\tau(a^\tau) + A_2^\tau(a^\tau) - 1.0) \geq 0.5 \). Hence, it follows that \( A_1^\tau(a^\tau) + A_2^\tau(a^\tau) - 1 \geq 0.5 \iff A_1^\tau(a^\tau) + A_2^\tau(a^\tau) \geq 1.5 \). Now, we do not know exactly the degrees of truth of \( A_1^\tau(a^\tau) \) and \( A_2^\tau(a^\tau) \), but they belong to \( \mathcal{N} \), so there are six possibilities:
Hence, we can think of a model satisfying \( a: (A_{1 \geq 0.5} \cap A_{2 \geq 1}) \cup (A_{1 \geq 0.75} \cap A_{2 \geq 0.75}) \cup (A_{1 \geq 0.75} \cap A_{2 \geq 1}) \cup (A_{1 \geq 1} \cap A_{2 \geq 0.5}) \cup (A_{1 \geq 1} \cap A_{2 \geq 0.75}) \cup (A_{1 \geq 1} \cap A_{2 \geq 1}) \).

The previous example shows that the disjunctions and conjunctions that are introduced in the reduction can be optimized by taking into account the following observations:

**Proposition 4.3.** Let \( B_1, B_2 \) be two non-fuzzy concepts such that \( B_1 \subseteq B_2 \). The following hold:

1. \( B_1 \cap B_1 \equiv B_1 \cup B_1 \equiv B_2 \).
2. \( B_1 \cap B_2 \cap B_2 \cap B_3 \).
3. \( B_1 \cap B_2 \equiv B_2 \cap B_3 \).
4. \( B_1 \cap B_2 \cap B_2 \cap B_3 \equiv B_2 \equiv B_3 \equiv B_m \).
5. \( B_1 \cap B_2 \cap B_3 \cap B_m \equiv B_1 \cap B_2 \cap B_3 \cap \ldots \cap B_m \).

**Proof.** Trivial. \( \square \)

**Example 4.4.** Consider the (non-fuzzy) assertion obtained in Example 4.2 as a result of the reduction. It can be seen that \( (A_{1 \geq 0.75} \cap A_{2 \geq 0.75}) \equiv (A_{1 \geq 0.75} \cap A_{2 \geq 1}) \) and that \( (A_{1 \geq 1} \cap A_{2 \geq 0.5}) \equiv (A_{1 \geq 1} \cap A_{2 \geq 1}) \). Consequently, the reduction of the axiom is satisfiable iff the following assertion is: \( a: (A_{1 \geq 0.5} \cap A_{2 \geq 1}) \cup (A_{1 \geq 0.75} \cap A_{2 \geq 0.75}) \cup (A_{1 \geq 0.75} \cap A_{2 \geq 1}) \cup (A_{1 \geq 1} \cap A_{2 \geq 0.5}) \cup (A_{1 \geq 1} \cap A_{2 \geq 0.75}) \cup (A_{1 \geq 1} \cap A_{2 \geq 1}) \).

Concept and role expressions are reduced using mapping \( \rho \), as shown in Table 6.

Mapping \( \rho \) deserves some comments. Firstly, it is interesting to remark that \( \rho(A_{\geq \beta}) = \neg A_{\leq \beta} \) is different to \( \rho(\neg A_{\geq \beta}) = \rho(A_{\leq 1 - \beta}) = \neg A_{\geq 1 - \beta} \). Then, due to the restrictions in the definition of the fuzzy KB, some expressions cannot appear during the process:

- Expressions of the form \( \rho(A_{\geq 0}) \) and \( \rho(A_{\leq 1}) \) cannot appear, because there exist some restrictions on the degree of the axioms in the fuzzy KB. The same also holds for \( \top, \bot, \triangledown \), and \( R_x \).
- Expressions of the form \( \rho(R, \leq \beta) \) can only appear in a negated role assertion.
- Expressions of the form \( \rho(U, \leq \beta) \) cannot appear either.

The case of qualified cardinality restrictions is more involved. We will use a partition is used to simulate the existence of \( m \) different individuals (the fillers of the cardinality restriction).

Let \( B_1, \ldots, B_m \) be atomic concepts. \( B_1, \ldots, B_m \) form a partition w.r.t. a fuzzy interpretation \( I \) iff the following conditions hold:

- \( \bigcup_{i=1}^{m} B_i^I = A^I \).
- \( B_i^I \cap B_j^I = \emptyset \) for \( 1 \leq i < j \leq m \).

For every expression of the form \( \rho( \geq m S.C. \beta \gamma ) \) that appear in the reduction process, we create \( m \) new atomic concepts \( B_1, \ldots, B_m \) such they form a partition w.r.t. \( I \). This is achieved by adding the following axioms:

- \( \top \subseteq B_1 \cup B_2 \cup \ldots \cup B_m \).
- \( B_i \cap B_j \subseteq \bot \), for \( i < j, 1 \leq i < j \leq m \).

**Proposition 4.5.** Let \( B_1, \ldots, B_m \) be non-empty atomic concepts forming a partition w.r.t. a fuzzy interpretation \( I \), and let \( b_i \) denote an individual such that \( b_i \in B_i^I \), for all \( i = 1, \ldots, m \). Then, \( b_1, \ldots, b_m \in A^I \) are pairwise different individuals.

**Proof.** By reduction to absurd. Assume on the contrary that there are two individuals \( b_i \) and \( b_j \) such that \( b_i = b_j \). By assumption, we have that \( b_i \in B_i^I \) and \( b_j \in B_j^I \). Using that \( b_i = b_j \) it follows that \( b_i \in B_j^I \). Since \( b_i \in B_i^I \) and \( b_j \in B_j^I \), \( b_i \in (B_i \cap B_j)^I \). Finally, using the condition of the partition \( B_i \cap B_j \subseteq \bot \), it follows that \( b_i^I \in \bot^I \), which is absurd. \( \square \)

**Proposition 4.6.** Let \( b_1, \ldots, b_m \in A^I \) be pairwise different individuals. Then, there exist atomic concepts \( B_1, \ldots, B_m \) forming a partition w.r.t. a fuzzy interpretation \( I \) such that \( b_i \in B_i^I \), for all \( i = 1, \ldots, m \).
Table 6
Mapping of concept and role expressions in fuzzy SROIQ.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(y)</th>
<th>(\rho(x,y))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T)</td>
<td>(\geq x)</td>
<td>(T)</td>
</tr>
<tr>
<td>(T)</td>
<td>(\leq y)</td>
<td>(y)</td>
</tr>
<tr>
<td>(\bot)</td>
<td>(\geq x)</td>
<td>(\bot)</td>
</tr>
<tr>
<td>(\bot)</td>
<td>(\leq y)</td>
<td>(\bot)</td>
</tr>
<tr>
<td>(A)</td>
<td>(\geq x)</td>
<td>(A_{\geq x})</td>
</tr>
<tr>
<td>(A)</td>
<td>(\leq y)</td>
<td>(-A_{\leq y})</td>
</tr>
<tr>
<td>(-C)</td>
<td>(\geq \gamma)</td>
<td>(\rho(C, \geq \gamma))</td>
</tr>
<tr>
<td>(-C)</td>
<td>(\leq \gamma)</td>
<td>(\rho(C, \geq \gamma))</td>
</tr>
<tr>
<td>(C \cap D)</td>
<td>(\geq \gamma)</td>
<td>(\rho(C \cap D, \geq \gamma))</td>
</tr>
<tr>
<td>(C \cap D)</td>
<td>(\leq \gamma)</td>
<td>(\rho(C \cap D, \geq \gamma))</td>
</tr>
<tr>
<td>(C \cup D)</td>
<td>(\geq \gamma)</td>
<td>(\rho(C \cup D, \geq \gamma))</td>
</tr>
<tr>
<td>(C \cup D)</td>
<td>(\leq \gamma)</td>
<td>(\rho(C \cup D, \geq \gamma))</td>
</tr>
<tr>
<td>(\exists R)</td>
<td>(\geq \gamma)</td>
<td>(\rho(\exists R, \geq \gamma))</td>
</tr>
<tr>
<td>(\exists R)</td>
<td>(\leq \gamma)</td>
<td>(\rho(\exists R, \geq \gamma))</td>
</tr>
<tr>
<td>(\forall R)</td>
<td>(\geq \gamma)</td>
<td>(\rho(\forall R, \geq \gamma))</td>
</tr>
<tr>
<td>(\forall R)</td>
<td>(\leq \gamma)</td>
<td>(\rho(\forall R, \geq \gamma))</td>
</tr>
<tr>
<td>({x_{1}, x_{2}, \ldots, x_{n}})</td>
<td>(\geq \gamma)</td>
<td>(\rho({x_{1}, x_{2}, \ldots, x_{n}}, \geq \gamma))</td>
</tr>
<tr>
<td>({x_{1}, x_{2}, \ldots, x_{n}})</td>
<td>(\leq \gamma)</td>
<td>(\rho({x_{1}, x_{2}, \ldots, x_{n}}, \leq \gamma))</td>
</tr>
</tbody>
</table>

Proof. It is trivial to assume the existence of \(m\) atomic concepts verifying \(\bigcup_{i=1}^{m} \{R_{i}^{2}\} = A^{2}\) and \(b_{i} \in B_{i}^{2}\). Since \(b_{i} \neq b_{j}\) for \(1 \leq i < j \leq m\), the condition \(B_{i}^{2} \cap B_{j}^{2} = \emptyset\) can also be satisfied. \(\Box\)

Then, \(\rho(\geq m S.C, \geq \gamma)\) is transformed into a conjunction of \(m\) expressions of the form \((\geq 1 \rho(S, \geq \gamma_{i}^{1}).(\rho(C, \geq \gamma_{i}^{2}) \cap B_{i}))\), each of them simulating one of the mutually different \(m\) fillers of the cardinality restriction, and in such a way that the degrees \(\gamma_{i}^{1}, \gamma_{i}^{2}\) satisfy the semantics of the constructor.

The reduction \(\rho(\geq m S.C, \leq \gamma)\) is based on the equivalence:

\[
\rho(\geq m S.C, \leq \gamma) \equiv \neg \rho(\geq m S.C, \geq \gamma)
\]

Finally, the reduction of \(\rho(\leq m S.C, \geq \gamma)\) is based on the equivalence:

\[
\rho(\leq m S.C, \leq \gamma) \equiv \neg (\rho(\leq m S.C, \geq \gamma))
\]

Axioms are reduced as in Table 7, where \(\kappa\) maps fuzzy ABox, TBox, and RBox axiom into non-fuzzy ABox, TBox, and RBox axioms, respectively.

The reader may be tempted to think that the reduction of a fuzzy GCI should take into account every pair \(\gamma_{1}, \gamma_{2} \in N^{+}\) such that \(\gamma_{1} \leq \gamma_{2} + 1 - \alpha\). However, the additional axioms are superfluous as the following example illustrates. The case of fuzzy RIA is similar.

Example 4.7. Consider a fuzzy GCI \(\langle A_{1} \sqsubseteq A_{2}, 0.5 \rangle\) and \(N = \{0.25, 0.5, 0.75, 1\}\). For every individual \(x\) of the interpretation domain, it follows that \(A_{1}^{2}(x) \geq A_{2}^{2}(x) \geq 0.5\) and thus \(1 - A_{1}^{2}(x) + A_{2}^{2}(x) \geq 0.5\). This introduces several possibilities:
It is easy to see that $A_1^x \supset A_2^y \iff 0.75$ implies that $A_1^y \supset A_2^z \iff 0.25$, and that $A_1^y \supset 1$ implies that $A_2^z \supset 0.5$. This restriction is hence equivalent to this couple of GCIs: $\rho(A_1, 0.75) \subseteq \rho(A_2, 0.25)$, and $\rho(A_1, 1) \subseteq \rho(A_2, 0.5)$.

As we see, we take $\gamma_1, \gamma_2 \in \mathbb{N}^+$ verifying $\gamma_1 = \gamma_2 + 1 - x = \gamma_2 + 0.5$.

Without loss of generality (see Proposition 3.4), we assume that negated role assertions, transitive and symmetric role axioms do not appear in the fuzzy KB. However, we do include the reduction of irreflexive role axioms because it is more efficient than using the equivalence in Proposition 3.4.

We note $\kappa(A)$ (resp. $\kappa(T), \kappa(R)$) the union of the reductions of every axiom in $A$ (resp. $T, R$). To be precise, the reduction of fuzzy GCIs and RIs should be noted as $\kappa(T, \mathcal{N})$, and the reduction of the fuzzy TBox and RBox as $\kappa(R, \mathcal{N})$, respectively. For the sake of simplicity we omit $\mathcal{N}$ since it is clear from the context.

Let crisp($\mathcal{K}$) denote the reduction of a fuzzy ontology $\mathcal{K}$. A fuzzy KB $\mathcal{K} = (A, T, R)$ is reduced into a KB crisp($\mathcal{K}$) = $(\kappa(A), T(\mathcal{N}) \cup \kappa(T), R(\mathcal{N}) \cup \kappa(R))$.

4.3. Examples

Now we will illustrate how the reduction works using two examples.

Example 4.8. Let us consider a fuzzy KB $\mathcal{K} = \{(a : \forall \alpha. (C \land D) \supset 0.75), (a, b) : R \supset 0.75), (b : \neg C \supset 0.75))\}$ and assume a set of degrees of truth $\mathcal{N} = \{0.25, 0.5, 0.75, 1\}$ ($n = 4$). Note that the TBox and the RBox are empty.

This fuzzy KB is clearly unsatisfiable. From the third assertion it follows that $C^\alpha(b^\beta) \supset 0.25$, and it can be seen that this implies that $(C \land D)^\alpha(b^\beta) = \max(C^\alpha + D^\beta - 1, 0) \leq 0.25$. But from the two former assertions it follows that every fuzzy interpretation $I$ has to satisfy $(C \land D)^\alpha(b^\beta) > 0.5$, which is a contradiction.

Now, let us compute the non-fuzzy representation of $\mathcal{K}$. Firstly, we create some new non-fuzzy atomic concepts associated to the set of atomic fuzzy concepts and some new non-fuzzy atomic roles associated to the set of atomic fuzzy roles:

- **New concepts:** $C_{0.25}, C_{0.25}, C_{0.25}, C_{0.5}, C_{0.5}, C_{0.75}, C_{0.75}, C_{1}, D_{0}, D_{0.25}, D_{0.25}, D_{0.5}, D_{0.5}, D_{0.75}, D_{0.75}, D_{1}$.

- **New roles:** $R_{0.25}, R_{0.5}, R_{0.75}, R_{1}$.

Now we create some new axioms to preserve the semantics of these elements:

- $T(\mathcal{N}) = \{C_{0.1} \subseteq C_{0.25}, C_{0.75}, C_{0.75} \subseteq C_{0.75}, C_{0.75} \subseteq C_{0.5}, C_{0.5} \subseteq C_{0.5}, C_{0.5} \subseteq C_{0.25}, C_{0.25} \subseteq C_{0.25}, C_{0.25} \subseteq C_{0.25}, D_{0}, D_{0.25}, D_{0.25}, D_{0.5}, D_{0.5}, D_{0.75}, D_{0.75}, D_{1}\}$.

- $R(\mathcal{N}) = \{R_{0.1} \subseteq R_{0.25}, R_{0.75} \subseteq R_{0.5}, R_{0.5} \subseteq R_{0.25}\}$.

### Table 7

Reduction of the axioms.

<table>
<thead>
<tr>
<th>$\kappa(a : C \supset a)$</th>
<th>$\kappa(a : C \supset b)$</th>
<th>$\kappa([a, b] : R \supset a)$</th>
<th>$\kappa([a, b] : R \supset b)$</th>
<th>$\kappa([a, b] : R \equiv [a, b])$</th>
<th>$\kappa(C \equiv D \supset a)$</th>
<th>$\kappa([R_1 \ldots R_n \supset R] \supset a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha, \rho(C, &gt; a)$</td>
<td>$\alpha, \rho(C, &gt; b)$</td>
<td>$\alpha, \rho(R, &gt; a)$</td>
<td>$\alpha, \rho(R, &lt; b)$</td>
<td>$a = b$</td>
<td>$\bigcup_{\gamma_1, \gamma_2} {\rho(C, &gt; \gamma_1) \subseteq \rho(D, &gt; \gamma_2)}$</td>
<td>$\bigcup_{\gamma_1, \ldots, \gamma_n} {\rho([R_1, \ldots, R_n] \supset \gamma_1) \subseteq \rho(R, \gamma_{n+1})}$</td>
</tr>
</tbody>
</table>

for every pair $\gamma_1, \gamma_2 \in \mathbb{N}^+$ such that $\gamma_1 > \gamma_2 + 1 - x$

for every combination $\gamma_1, \ldots, \gamma_{n+1} \in \mathbb{N}^+$ such that $\gamma_1 + \ldots + \gamma_{n+1} + n - x$

- $\kappa(add(S_1) \supset \kappa(re(f(R))$ |

- $\kappa(\text{inc}(S))$  | $\text{inc}(\rho(S, > 1))$

- $\kappa(\text{as}(S))$  | $\text{as}(\rho(S, > 0))$
Now we are ready to compute $\kappa(A)$, including the reduction of the three fuzzy assertions in the fuzzy KB, that is:

- $\kappa([a, b] : R \geq 0.75) = (a, b) : \rho(R, \geq 0.75) = (a, b) : R \geq 0.75$.
- $\kappa([b : -C \geq 0.75]) = b : \rho(-C, \geq 0.75) = b : -C \geq 0.25$.
- $\kappa([a : \forall R. (C \cap D) \geq 0.75]) = a : \rho(\forall R. (C \cap D) \geq 0.25) \cap \forall \rho(R, \geq 0.75)$.

where:

- $\rho(R, \geq 0.75) = R \geq 0.5$.
- $\rho(C \cap D, \geq 0.25) = (C \geq 0.25 \cap D) \geq 0.25$.
- $\rho(R, \geq 0.75) = R \geq 0.75$.
- $\rho(C \cap D, \geq 0.5) = (C \geq 0.5 \cap D) \geq 0.5$.
- $\rho(R, \geq 1) = R \geq 1$.
- $\rho(C \cap D, \geq 0.75) = (C \geq 0.75 \cap D) \geq 0.75$.

It can be seen that the (non-fuzzy) KB $\text{crisp}(\kappa) = \langle \kappa(A), T(\mathcal{A}), R(\mathcal{A}) \rangle$ is unsatisfiable.

Next we illustrate how to reduce cardinality restrictions, which are more involved.

**Example 4.9.** Let us consider a fuzzy KB $\mathcal{K} = \{\{(a : \leq 15.5 \geq 0.75), (a, b) : S \geq 0.75, (a, c) : S \geq 0.75, (b : C \geq 0.75), (c : C \geq 0.75), (b \neq c)\} \}$ and assume a set of degrees of truth $\mathcal{N} = \{0, 0.25, 0.5, 0.75, 1\} (n = 4)$. Note that the TBox and the RBox are empty.

This fuzzy KB is clearly unsatisfiable. From the first assertion it follows that $\min\{S^f(a, b) \cap C^f(b), S^f(a, c) \cap C^f(c)\} \Rightarrow (b = c) \geq 0.75$. This is true in two cases:

- $b = c$, or
- $\min\{S^f(a, b) \cap C^f(b), S^f(a, c) \cap C^f(c)\} \leq 0.25$.

The first possibility is clearly in contradiction with the last assertion of the fuzzy KB. Moreover, assertions 2, 3, 4 and 5 imply that $\min\{S^f(a, b) \cap C^f(b), S^f(a, c) \cap C^f(c)\} = \min\{0.75 \cap 0.75, 0.75 \cap 0.75\} = 0.5$, which is in contradiction with the second possibility. Hence, the fuzzy KB is unsatisfiable.

Now, let us compute the non-fuzzy representation of $\mathcal{K}$. Firstly, we create some new non-fuzzy atomic concepts associated to the set of atomic fuzzy concepts and some new non-fuzzy atomic roles associated to the set of atomic fuzzy roles:

- **New concepts:** $C_{\geq 0.75}, C_{\geq 0.25}, C_{\geq 0.5}, C_{\geq 0.5}, C_{\geq 0.75}, C_{\geq 1.1}$.
- **New roles:** $S_{\geq 0.25}, S_{\geq 0.5}, S_{\geq 0.75}, S_{\geq 1}$.

Now we create some new axioms to preserve the semantics of these elements:

- $T(\mathcal{A}) = \{C_{\geq 0.01} \subseteq C_{\geq 0.75}, C_{\geq 0.75} \subseteq C_{\geq 0.75}, C_{\geq 0.75} \subseteq C_{\geq 0.5}, C_{\geq 0.5} \subseteq C_{\geq 0.25}, C_{\geq 0.25} \subseteq C_{\geq 0.25}, C_{\geq 0.25} \subseteq C_{\geq 0.5})\}$.
- $R(\mathcal{A}) = \{S_{\geq 0.01} \subseteq S_{\geq 0.75}, S_{\geq 0.75} \subseteq S_{\geq 0.75}, S_{\geq 0.75} \subseteq S_{\geq 0.25}\}$.

Now we are ready to compute $\kappa(A)$, including the reduction of the six fuzzy assertions in the fuzzy KB, that is:

$$
\kappa(a : (\leq 15.5 \geq 0.75)) = a : \rho(\leq 25C, \geq 0.75) = a : \rho(\geq 25C, \leq 0.25) = \left(\begin{array}{c}
(\exists \rho(S, \geq 0.5) \land (B_1 \land S \geq 0.5) \land (B_2 \land S \geq 0.5)) \\
\exists \rho(S, \geq 0.5) \land (B_1 \land S \geq 0.5) \\
\exists \rho(S, \geq 0.5) \land (B_2 \land S \geq 0.5) \\
(\exists \rho(S, \geq 0.5) \land (B_1 \land S \geq 0.5) \land (B_2 \land S \geq 0.5)) \\
(\exists \rho(S, \geq 0.5) \land (B_1 \land S \geq 0.5) \land (B_2 \land S \geq 0.5)) \\
(\exists \rho(S, \geq 0.5) \land (B_1 \land S \geq 0.5) \land (B_2 \land S \geq 0.5)) \\
(\exists \rho(S, \geq 0.5) \land (B_1 \land S \geq 0.5) \land (B_2 \land S \geq 0.5)) \\
(\exists \rho(S, \geq 0.5) \land (B_1 \land S \geq 0.5) \land (B_2 \land S \geq 0.5)) \\
(\exists \rho(S, \geq 0.5) \land (B_1 \land S \geq 0.5) \land (B_2 \land S \geq 0.5))
\end{array}\right)
$$

The reduction of this concept expression also introduces two new atomic concepts $B_1, B_2$ together with the axioms:

$$
\kappa([a, b] : S \geq 0.75) = (a, b) : \rho(S, \geq 0.75) = (a, b) : S \geq 0.75.
$$

$$
\kappa([b : C \geq 0.75]) = b : \rho(C, \geq 0.75) = b : C \geq 0.75.
$$

$$
\kappa([a, c] : S \geq 0.75) = (a, b) : \rho(S, \geq 0.75) = (a, c) : S \geq 0.75.
$$

$$
\kappa([c : C \geq 0.75]) = b : \rho(C, \geq 0.75) = c : C \geq 0.75.
$$

$$
\kappa(b \neq c) = b \neq c.
$$
It can be seen that the (non-fuzzy) KB \( \text{crisp}(\mathcal{K}) = (\kappa(\mathcal{A}), T(\mathcal{A}), R(\mathcal{R})) \) is unsatisfiable.

4.4. Properties of the reduction

Firstly, we highlight that the reduction preserves simplicity of the roles and regularity of the RIAs.

Correctness. The reduction is reasoning preserving and, since satisfiability testing in classical SROIQ is decidable \([30]\) and the mapping is finite it follows that:

**Theorem 4.10.** The satisfiability problem in \( \mathbb{L}_n^{\text{SROIQ}} \) with truth space \( \{0, 1, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{(n-1)}{n}, 1\} (n \in \mathbb{N}) \) is decidable. Furthermore, a \( \mathbb{L}_n^{\text{SROIQ}} \) fuzzy KB \( \mathcal{K} \) is satisfiable iff \( \text{crisp}(\mathcal{K}) \) is satisfiable.

**Proof.** See Appendix. \( \square \)

**Complexity.** The depth of a fuzzy concept is inductively defined as follows:

1. \( \text{depth}(A) = \text{depth}(\top) = \text{depth}(\bot) = \text{depth}(\exists S, \text{Self}) = \text{depth}(\{x_1, o_1, \ldots, x_m, o_m\}) = 1. \)
2. \( \text{depth}(\neg C) = \text{depth}(\forall R, C) = \text{depth}(\exists R, C) = \text{depth}(\geq \equiv nS, C) = 1 + \text{depth}(C). \)
3. \( \text{depth}(C \cap D) = \text{depth}(C \cup D) = 1 + \max \{\text{depth}(C), \text{depth}(D)\}. \)

The depth of a non-fuzzy concept is defined analogously. It is easy to see that:

- A concept \( C \) without number restrictions of depth \( k \) transforms in the worst case into an expression of size \( \mathcal{O}(|C||\mathcal{N}|^k) \). For instance, for \( C \) being \( \forall R. (\forall Q, A) \), we get an expression of size \( \mathcal{O}(|C||\mathcal{N}|^3) \) \((k = 3)\).
- A concept \( C \) with number restrictions of depth \( k \) transforms in the worst case into an expression of size \( \mathcal{O}(m^{k-1}|C| |\mathcal{N}|^{k-1}^m) \), where \( m \) is the maximal number restriction occurring in \( C \). For instance, for \( C \) being \( (\geq mR. (\geq mQPA)) \), we get an expression of size \( \mathcal{O}(m^2|C| |\mathcal{N}|^{2^m}) \) \((k = 3, m = \max(m_1, m_2)\).

In order to preserve the semantics of the new atomic concept and roles, we are also introducing some new non-fuzzy axioms:

\[
|T(\mathcal{A})| = (2 \cdot (|\mathcal{A}| - 1) - 1) \cdot |A| \\
|R(\mathcal{R})| = (2 \cdot (|\mathcal{R}| - 1) - 1) \cdot |R|
\]

- The reduction of qualified cardinality restrictions \( \rho(\geq mS, C, \lor \gamma) \) also introduces \( \binom{m}{2} + 1 \) GCIs.
- Most of the axioms of the fuzzy KB generate one non-fuzzy axiom, but some of them (fuzzy GCIs and fuzzy RIAs) generate several non-fuzzy axioms:

\[
|\kappa(T)| \leq 2 \cdot (|\mathcal{N}| - 1) \cdot |T| \\
|\kappa(R)| \leq 2 \cdot (|\mathcal{R}| - 1) \cdot |R|
\]

All in all, the size of the resulting KB is \( \mathcal{O}(|\mathcal{K}||\mathcal{N}|^k) \) in case no number restriction occurs in \( \mathcal{K} \), where \( k \) is the maximal depth of the concepts appearing in the fuzzy KB, while otherwise is \( \mathcal{O}(m^{k-1}|\mathcal{K}| |\mathcal{N}|^{k-1}^m) \), where \( m \) is the maximal number restriction occurring in \( \mathcal{K} \).

We recall that under Zadeh fuzzy logic, the size of the resulting KB is \( \mathcal{O}(|\mathcal{K}|^k) \) \([61, 7, 5]\). In our case we need to generate more and more complex axioms, because we cannot infer the exact values of the elements which take part of a complex concept, so we need to build disjunctions or conjunctions over all possible degrees of truth.

**Modularity.** An interesting property of the procedure is that the reduction of an ontology can be reused when adding new axioms and only the reduction of the new axioms has to be included. From an implementation point of view, this property allows to compute the reduction of the ontology off-line and update \( \kappa(\mathcal{K}) \) incrementally.

**Theorem 4.11.** Let \( \mathcal{K} \) be a \( \mathbb{L}_n^{\text{SROIQ}} \) fuzzy knowledge base involving a set of fuzzy atomic concepts \( \mathcal{A} \) and a set of atomic roles \( \mathcal{R} \); let \( \mathcal{N} = \{0, 1, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{(n-1)}{n}, 1\} (n \in \mathbb{N}) \); and let \( \tau \) be a \( \mathbb{L}_n^{\text{SROIQ}} \) axiom such that:

1. for every atomic concept \( A \) which appears in \( \tau \), \( A \in \mathcal{A} \).
2. for every atomic role \( R_A \) which appears in \( \tau \), \( R_A \in \mathcal{R} \).
3. if \( \gamma \) appears in \( \tau \), then \( \gamma \in \mathcal{N} \).

Then, \( \text{crisp}(\mathcal{K} \cup \tau) = \text{crisp}(\mathcal{K}) \cup \kappa(\tau) \).
Proof. The proof is trivial from the following observations:

- Every axiom is reduced to a combination of new non-fuzzy elements.
- New elements depend on fuzzy atomic concepts, fuzzy roles and the membership degrees appearing in the fuzzy KB.
- \( \tau \) does not introduce atomic concepts, atomic roles nor new membership degrees with respect to the fuzzy KB.
- Every axiom is mapped independently from the others. □

The theorem assumes that the set of possible degrees in the language is restricted and that the basic vocabulary (concepts and roles) is fully expressed in the ontology and does not change often. These are reasonable assumptions because ontologies do not usually change once that their development has finished. Moreover, we have assumed a fixed set of the degrees of truth \( N \).

5. Related work

Since the first work of J. Yen in 1991 [65], an important number of fuzzy extensions to DLs can be found in the literature [41]. We would also like to stress the existence of fuzzy rough DLs [13,33,34] (which extend rough DLs [40,19,51,21,35]) and fuzzy possibilistic DLs [6].

In this section we will concentrate on the state of the art on the fuzzy logics considered in the framework of fuzzy DLs which are different from Zadeh fuzzy logic, and in the representation of fuzzy DLs using non-fuzzy DLs.

5.1. Fuzzy logics in fuzzy DLs

While most of the works restrict themselves to Zadeh fuzzy logic, a few other works consider \( \text{Łukasiewicz} \) fuzzy logic. Straccia and coworkers [58,62,59,12] propose a reasoning solution, which is based on a mixture of tableau rules and Mixed Integer Linear Programming (MILP) optimization problems. These works are implemented in the fuzzyDL reasoner [11]. Habiballa [25] considers a fuzzy extension of \( \text{ALC} \) extended with role negation, top role and bottom role, presenting a novel reasoning algorithm based on resolution, as well as an implementation (GERDS). Another implementation based on resolution (YADLR) has been recently presented [38].

A proposal for a product t-norm-based fuzzy DL has also been presented [9], using Product logic but replacing Gödel negation with \( \text{Łukasiewicz} \) negation.

A Gödel fuzzy DL has been presented in [8]. Previously, [7] considered Gödel implication, but only in the semantics of GCIs and RIA.

There are also some attempts to reason with arbitrary continuous t-norms. Hájek reported a reasoning algorithm based on a reduction to fuzzy propositional logic [27]. The authors of the present paper have also recently presented a reasoning algorithm for fuzzy DLs under arbitrary continuous t-norms extended with \( \text{Łukasiewicz} \) negation, based on a combination of tableau rules and Mixed Integer Non Linear Programming (MINLP) optimization problems [10]. Both of these works are restricted to the witnessed models of fuzzy \( \text{ALC} \) without fuzzy GCIs.

5.2. Non-fuzzy representations for fuzzy DLs

The first effort in this direction is a reasoning preserving procedure for fuzzy \( \text{ALCH} \) [61]. A similar work from him considers fuzzy \( \text{ALC} \) with truth values taken from an uncertainty lattice [57], therefore supporting quantitative reasoning (by using the interval \([0,1]\)) and qualitative reasoning (e.g. by relying on a set \{false, likelyfalse, unknown, likelytrue, true\}). Bobillo et al. widened the former work of Straccia to \( \text{SHOIN} \) and allowed fuzzy GCIs, but with a semantics given by KD implication [4]. Stoilos et al. extended this work and considered the reduction of an extension of fuzzy \( \text{SHOIN} \) with additional role axioms: general RIA, reflexive, asymmetric and role disjointness axioms [53]. It is not a reduction of fuzzy \( \text{SROIQ} \) (not even \( \text{SROIN} \)) because they do not show how to reduce the universal role, qualified cardinality restrictions, local reflexivity concepts in expressions of the form \( \rho[\text{Self}, \psi] \) nor negative role assertions. Moreover, GCIs and RIA are forced to be either true or false. Bobillo et al. extended this work providing a non-fuzzy representation of full \( \text{SROIQ} \) with fuzzy GCIs and RIA [7].

A different approach considers a family of fuzzy DLs using \( \alpha \)-cuts as atomic concept and roles [43]. The approach is slightly different to ours because, in general, these logics need their own decision procedures. However, the authors have shown how to reduce a fuzzy \( \text{ALCQ} \) ABox [42] and a fuzzy \( \text{ALCH} \) concept [36] to their non-fuzzy versions. Nevertheless, both of these works assume an empty TBox. Finally, [20] combines possibilistic and fuzzy logics in the context of Description Logics (more concretely, in \( \text{ALCIN}(\circ) \)). Interestingly, they also propose to represent every fuzzy set using two sets (its support and its core) and comment the possibility of using more sets, in order to have a more refined representation. Although for some applications this representation may be enough, there is a loss of information that does not occur in our approach.

All this previous work has been restricted to Zadeh fuzzy logic, with the exception of a non-fuzzy representation for Gödel fuzzy SROIQ [8], and of a previous version of the current paper, which considers L-ALCHOI [14].

Finally, non-fuzzy representations for two components of fuzzy DLs (which are independent of the particular logic) have been proposed. [5] considers the reduction of fuzzy concrete domains, whereas [8] deals with modified fuzzy concepts and roles.

6. Conclusions and future work

In this paper we have shown the decidability of a fuzzy extension of SROIQ under the semantics of finitely many-valued Łukasiewicz fuzzy logic, assuming a fixed set of allowed degrees of truth. We have provided a reasoning algorithm based on a reduction to SROIQ. Together with the non-fuzzy representation of fuzzy concrete domains proposed in [5], this means the possibility to reason with Łukasiewicz fuzzy OWL 2.

Providing non-fuzzy representations for fuzzy DLs means an important step towards the possibility of dealing with imprecise and vague knowledge in DLs, since it relies on existing languages and tools. This approach has several advantages:

- We may continue using existing DL reasoners, which is important because current fuzzy DL reasoners cannot support a fuzzy extension of OWL 2 under Łukasiewicz fuzzy logic (FuzzyDL supports fuzzy OWL-Lite so far).
- Our work is more general than previous approaches which provide non-fuzzy representations of fuzzy DLs under Zadeh fuzzy logic. However, from a practical point of view, the size of the resulting KB is much more complex in this case, so the practical feasibility of this approach has to be empirically verified.

In future work we plan to implement the proposed reduction, studying if it can be optimized in some particular common situations.

Acknowledgements

The authors thank Juan Gómez-Romero and the anonymous referees for several corrections in an earlier version of this paper.

Appendix A. Proof of Theorem 4.10

Proof. We will show the proof for the only-if direction. From $\mathcal{K}$ is satisfiable we know that there is a fuzzy interpretation $\mathcal{I} = \{A^\mathcal{I}, \tau\}$ satisfying every axiom in $\mathcal{K}$. Now, it is possible to build a (non-fuzzy) interpretation $\mathcal{I}_c = \{A^c, \tau\}$ in the following way:

- $A^c = A^\mathcal{I}$.
- $a^c = a^\mathcal{I}$, for all $a \in A^\mathcal{I}$.
- $A^c_{A} = \{x \in A^\mathcal{I} | A^\mathcal{I}(x) \geq \alpha\}$, for each $A \in A$ and $\alpha \in N \setminus \{0\}$.
- $A^c_{A\beta} = \{x \in A^\mathcal{I} | A^\mathcal{I}(x) > \beta\}$, for each $A \in A$, $\beta \in N \setminus \{1\}$.
- $R^c_{A\alpha} = \{x, y \in A^\mathcal{I} \times A^\mathcal{I} | R^\mathcal{I}_A(x, y) \geq \alpha\}$, for each $R_A \in R$, $\alpha \in N \setminus \{0\}$.
- $R^c_{A\beta} = \{x, y \in A^\mathcal{I} \times A^\mathcal{I} | R^\mathcal{I}_A(x, y) > \beta\}$, for each $R_A \in R$, $\beta \in N \setminus \{1\}$.

Now, it can be shown that $\mathcal{I}_c$ satisfies every axiom in crisp($\mathcal{K}$). For every axiom $\tau \in \mathcal{K}$, there are several cases:

1. $\tau$ is an inequality assertion. Assume that $\mathcal{I} \models (a \neq b)$. Then, $a^\mathcal{I} \neq b^\mathcal{I}$. By definition of $\mathcal{I}_c$, $a^c \neq b^c$, so $\mathcal{I}_c \models (a \neq b)$.
2. $\tau$ is an equality assertion. Assume that $\mathcal{I} \models (a = b)$. Then, $a^\mathcal{I} = b^\mathcal{I}$. By definition of $\mathcal{I}_c$, $a^c = b^c$, so $\mathcal{I}_c \models (a = b)$.
3. $\tau$ is a role assertion. Assume that $\mathcal{I} \models (a, b) : R \bowtie \gamma$. We show, by induction on the structure of roles, that $\mathcal{I}_c \models \kappa((a, b) : R \bowtie \gamma)$.
   - Atomic role. Assume that $\mathcal{I} \models (a, b) : R_A \bowtie \alpha$. Then, $R^\mathcal{I}_A(a^\mathcal{I}, b^\mathcal{I}) \bowtie \alpha$. By definition of $\mathcal{I}_c$, it follows that $(a^c, b^c) \in R^c_{A\alpha}$. By definition of $\rho$, $(a^c, b^c) \in (\rho(R_A, \bowtie \alpha))^{c}$ $\iff \mathcal{I}_c \models (a, b) : \rho(R_A, \bowtie \alpha)$ $\iff \mathcal{I}_c \models \kappa((a, b) : R_A \bowtie \alpha)$.
   - Inverse role. Assume that $\mathcal{I} \models (a, b) : R^{-1} \bowtie \gamma$. Then, $R^\mathcal{I}_A(b^\mathcal{I}, a^\mathcal{I}) \bowtie \gamma$. By induction hypothesis, $(b^c, a^c) \in (\rho(R, \bowtie \gamma))^{c}$. Consequently, $(a^c, b^c) \in ((\rho(R, \bowtie \gamma))^{c})^{-1} \iff \mathcal{I}_c \models (a, b) \in \rho(R, \bowtie \gamma)^{-1}$ $\iff \mathcal{I}_c \models \kappa((a, b) : R^{-1} \bowtie \gamma)$.
4. Atomic concept. Assume that $I := \langle a : A \supseteq \alpha \rangle$. Then, $U^C(a^C, b^C) = 1 \supseteq \gamma$. By definition of $I_C^C$, it follows that $(a^C, b^C) \in A^C \times A^C$ and consequently $(a^C, b^C) \in U^C \iff (a^C, b^C) \in (\rho(U, \supseteq \alpha))^C \iff I_C = C(a, b) : \rho(U, \supseteq \alpha) \iff I_C = C(a, b) : \rho(U, \supseteq \alpha) \iff I_C = C(a, b) : \rho(U, \supseteq \alpha)$. The case $I := \langle a, b : U \not\supseteq \beta \rangle$ is similar.

4.1. We know that they certainly belong to $I_C$. Equivalently (and using Proposition 4.3), this can be simplified to $I_C = C(a, b) : \rho(U, \supseteq \alpha) \iff I_C = C(a, b) : \rho(U, \supseteq \alpha) \iff I_C = C(a, b) : \rho(U, \supseteq \alpha)$. Assume that $I := \langle a : A \not\supseteq \alpha \rangle$. This is equivalent to say that $I_C = C(a, b) : \rho(U, \supseteq \alpha) \iff I_C = C(a, b) : \rho(U, \supseteq \alpha)$. This case is similar to universal quantification. Assume that $I := \langle a : C \cap D \not\supseteq \beta \rangle$. Then, $\inf_{b \in A^C} \{R^C(a, b) \supseteq D^C(b)\} \not\supseteq \alpha$. Hence, for an arbitrary individual $b \in A^C$ it follows that $R^C(a, b) \supseteq D^C(b)$ holds. Now, one of the following conditions holds:

4. Concept disjunction. This case is similar to concept conjunction. Assume that $I := \langle a : C \cap D \not\supseteq \beta \rangle$. Then, $\sup_{b \in A^C} \{R^C(a, b) \supseteq D^C(b)\} \not\supseteq \alpha$. This is equivalent to say that $a^C \in (\rho(C, \supseteq \alpha) \cup \rho(D, \supseteq \alpha)) \iff I_C = \kappa(a : C \cap D \not\supseteq \beta)$. In the case $I := \langle a : C \cap D \not\supseteq \beta \rangle$ we use the equivalence $C \cap D \equiv \neg(C \cap \neg D)$ and consider $I := \langle a : C \cap D \not\supseteq \beta \rangle$. Note that condition $b$ is equivalent to $R^C(a, b) \supseteq C^C(b) \iff \alpha$. This makes the Lukasiewicz implication equal to $1 \supseteq \alpha$, or

4. Universal quantification. Assume that $I := \langle a : \forall R \supseteq \alpha \rangle$. Then, $\inf_{b \in A^C} \{R^C(a, b) \supseteq D^C(b)\} \not\supseteq \alpha$. Due to the WMP, there exists an individual $b \in A^C$ such that $R^C(a, b) \supseteq C^C(b) \not\supseteq \alpha \iff R^C(a, b) \supseteq C^C(b) \not\supseteq \alpha \iff R^C(a, b) \supseteq C^C(b) \not\supseteq \alpha \iff R^C(a, b) \supseteq C^C(b) \not\supseteq \alpha$. Note that condition $b$ is equivalent to $R^C(a, b) \supseteq C^C(b) \iff \alpha$. This makes the Lukasiewicz implication equal to $1 \supseteq \alpha$, or

4. Existential quantification. This case is similar to universal quantification. Assume that $I := \langle a : \exists R \supseteq \alpha \rangle$. Then, $\sup_{b \in A^C} \{R^C(a, b) \cap C^C(b)\} \not\supseteq \alpha$. Due to the WMP, there exists an individual $b \in A^C$ such that $R^C(a, b) \cap C^C(b) \not\supseteq \alpha \iff R^C(a, b) \cap C^C(b) \not\supseteq \alpha \iff R^C(a, b) \cap C^C(b) \not\supseteq \alpha \iff R^C(a, b) \cap C^C(b) \not\supseteq \alpha$. This is exactly the semantics of $a^C \in (\rho(R, \supseteq \gamma))^C \iff I_C = \kappa(a : \exists R \supseteq \gamma)$.

At-most qualified number restriction. Assume \( I \models \{ a: (\forall m \in \mathbb{C}) \geq a \}. \) Then, \( \sup_{b_1, \ldots, b_m \in A^t} \{ \min_{b_1, \ldots, b_m} \{ S^t(a^t, b^t) \cap C^t(b) \} \} \geq a \). Note that \( \sup_{b_1, \ldots, b_m \in A^t} \{ \min_{b_1, \ldots, b_m} \{ S^t(a^t, b^t) \cap C^t(b) \} \} = 0 \), which is not possible because by definition \( \forall \in \mathbb{C} \). Hence, \( \sup_{b_1, \ldots, b_m \in A^t} \{ \min_{b_1, \ldots, b_m} \{ S^t(a^t, b^t) \cap C^t(b) \} \} > 1 \) (which means that the elements are pairwise different).

Thanks to the WMP, for some pairwise different elements \( b_1, \ldots, b_m \in A^t \), \( \min_{b_1, \ldots, b_m} \{ S^t(a^t, b^t) \cap C^t(b) \} = 1 \). This means that every \( b_i \) satisfies \( S^t(a^t, b^t) \cap C^t(b) \). Similarly as with the conjunction, this means that \( S^t(a^t, b^t) \cap C^t(b) \), for some \( \gamma_i, \gamma_j \in \mathbb{N}^+ \) such that \( \gamma_i + \gamma_j + \gamma + 1 \geq a \). Furthermore, thanks to Proposition 4.6, we know that \( b_i \in B_i \), for some new concepts \( B_1, \ldots, B_m \) forming a partition w.r.t. \( I \)."
10. \( \tau \) is an asymmetry role axiom. Assume that \( \mathcal{I} \models \text{asy}(S) \). Then, \( \forall x, y \in A^\tau \), if \( S^\tau(x, y) > 0 \) then \( S^\tau(y, x) = 0 \). By induction, \( \forall x, y \in A^\tau \), if \( (x, y) \in (\rho(S, > 0))^\tau \) then \( (y, x) \in (\rho(S, < 0))^\tau \). Consequently, \( \mathcal{I} \models \text{asy}(\rho(S, > 0)) \).

The proof for the converse can be obtained using similar arguments: from a classical interpretation we build a fuzzy interpretation. There is only one point which is worth mentioning. If \( \text{crispic}(\mathcal{I}) \) is satisfiable, it is not possible (due to the axioms in \( T(N_\mathcal{I}) \)) to have an individual \( a \) such that \( \alpha^c(a) \in (A^\gamma, >)^\tau \) and \( \alpha^c(S) \notin (A^\gamma, <)^\tau \) with \( \gamma_2 \leq \gamma_1 \), so for every individual \( a \) we can compute the maximum value \( x \) such that \( \alpha^c(a, >) \) holds, or the maximum value \( \beta \) such that \( \alpha^c(a, >) \) holds, and use these values in the construction of the fuzzy interpretation. The case for roles in \( R(N_\mathcal{I}) \) is similar. \( \square \)

References


